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A time-dependent approach to the total scattering cross section

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Abstract. A method is developed, within the framework of time-dependent scattering theory, of proving finiteness at almost all energies of the total cross section for scattering by a wide class of potentials which roughly decrease more rapidly than r^{-2} at infinity, but which may have arbitrary local singularities.

Explicit bounds are obtained for weighted averages of the cross section over a range of energies; in particular the estimates for high-energy behaviour of cross sections are almost the best possible. Our method applies to a wide class of Hamiltonians, including the non-relativistic Schrödinger Hamiltonian and the Hamiltonian of a relativistic particle in a potential, and may be extended to many-particle systems. For the Schrödinger equation, our estimate of high-energy behaviour of cross sections with singular potentials is practically equivalent to the Froissart bound. There is a bound for potentials of finite range R which is independent of the coupling constant and of the form of the potential, and which, for large R, resembles the classical result.

1. Introduction

The most important quantity in scattering theory is the scattering cross section, which is directly related to experimental observation. It is therefore of considerable interest to look for methods enabling one to make rigorous statements about the cross section. Typically, such an approach includes two steps: (1) To obtain general expressions for the cross section in terms of the basic objects of the theory, which are the scattering operator in the Hilbert space approach or the non-normalisable solutions of the time-independent Schrödinger equation in the eigenfunction approach; (2) To use these expressions for deriving properties of the cross section under suitable assumptions on the interaction. In the Hilbert space approach, this means studying the properties of the on-shell S-matrix, or studying the behaviour at large distances of solutions of a partial differential equation in the eigenfunction approach.

In the present paper we shall adopt the Hilbert space method, which we believe to be more fundamental and which is based on the physically transparent formulation of the asymptotic condition in the time-dependent form (Jauch 1958). Step 1, to express the cross section in terms of the scattering operator, is relatively easy to carry out. The derivation is essentially time-dependent and will be briefly described in § 2. Step 2, the study of the scattering operator, is usually based on the equations of stationary state scattering theory, which are obtained as a consequence of the asymptotic condition. The

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stationary method leads to quite detailed information about the cross section but has the drawback of being technically rather involved. It involves estimating the behaviour of the resolvent $(H-z)^{-1}$ of the Hamiltonian H (or equivalently of the Green function $G_z(\boldsymbol{x}, \boldsymbol{y})$) as z approaches the positive real axis. These resolvent estimates are used to prove asymptotic completeness of the wave operators, which leads to an expression for the on-shell S-matrix in terms of the resolvent. This in turn can be used to prove, for example, the finiteness of the total cross section, its continuity as a function of the energy and to establish its high and low energy behaviour as well as properties of the scattering amplitude; see chapters 7, 10, 12 and 16 of Amrein *et al* (1977, referred to as A).

In view of the complexity of the stationary approach, we have developed a new and elementary method for deriving properties of the total cross section. This method not only dispenses with resolvent estimates but also does not involve any explicit expression for the on-shell S-matrix. The basic technical input are certain Cook–Hack type estimates in the Hilbert space $\mathscr{B}_2$ of Hilbert–Schmidt operators. Despite its simplicity this approach leads to interesting bounds on the cross section, both for the two-body and for the N-body problem. Our approach is in the spirit of a recent trend in scattering theory which is to work in the time-dependent framework (Amrein and Georgescu 1973, Pearson 1975b, Deift and Simon 1976, Enss 1977, Simon 1977, 1978, Sinha 1977) rather than using the resolvent and which has led to the remarkable time-dependent proof by Enss (1978) of strong asymptotic completeness (including the absence of the singularly continuous spectrum of H) in two-body potential scattering. It seems to us that these time-dependent arguments are more transparent than the stationary methods and will considerably simplify the teaching of scattering theory.

We consider two-body potential scattering for potentials that satisfy

$$\int_{|\boldsymbol{x}|\geqslant M} \mathrm{d}^3x(1+|\boldsymbol{x}|)^{2\upsilon}|V(\boldsymbol{x})|^2<\infty \tag{1}$$

for some $M \geqslant 0$ and some $\upsilon > \frac{1}{2}$. Notice that (1) restricts only the behaviour of V near infinity and is verified for example if $|V(\boldsymbol{x})| \leqslant |\boldsymbol{x}|^{-\mu}$ for $|\boldsymbol{x}| \geqslant M$ $(0<M<\infty)$ and some $\mu>2$. Only very weak hypotheses on the local behaviour of the potential will be made. In particular V may have an *arbitrary* singularity at $\boldsymbol{x}=0$.

We denote by $\bar{\sigma}(\lambda)$ the average over all initial directions of the total cross section at energy λ, in the centre-of-mass frame. It will be shown that $\bar{\sigma}(\lambda)$ is locally integrable, i.e. that

$$\int_0^\infty \mathrm{d}\lambda\, h(\lambda)\bar{\sigma}(\lambda)<\infty \tag{2}$$

for suitable weight functions h. This implies that $\bar{\sigma}(\lambda)$ is finite for almost all λ, and that its average

$$\frac{1}{2\epsilon}\int_{E-\epsilon}^{E+\epsilon} \mathrm{d}\lambda\, \bar{\sigma}(\lambda)$$

over an interval of length 2ϵ around E is a continuous function of E.

Explicit bounds on the integral in (2) in terms of h, of the weighted L^2-norm (1) of the potential V and of the cut-off parameter M will be given. If the integral in (1) is zero for some M, i.e. if V is of finite range, then the bound depends only on h and M, irrespective of the form of V for $|\boldsymbol{x}|<M$ and of the coupling constant. This is somewhat

analogous to the classical case where $\sigma(\lambda)$ is equal to πM^2, where M is the range of V (Newton 1966, chapter 5). By suitably choosing the high-energy behaviour of the weight function h, (2) also implies bounds on the high-energy behaviour of the total cross section. Roughly, these bounds are equivalent to $\bar{\sigma}(\lambda) < c\lambda^{-1+\epsilon}$ for each $\epsilon > 0$ if $M = 0$ in (1) and $\bar{\sigma}(\lambda) < c\lambda^{\epsilon}$ for each $\epsilon > 0$ if the potential has strong local singularities.

We are aware of the following papers dealing with the finiteness of the cross section for the two-body problem. Green and Lanford (1960) and Misra *et al* (1963), by estimating the phase shifts, derive finiteness for non-singular spherically-symmetric potentials which are $O(r^{-2-\epsilon})$, $\epsilon > 0$, at infinity; Villarroel (1970) obtains finiteness for essentially the same class of potentials as ours by working with the continuum eigenfunctions of the Hamiltonian and postulating a radiation condition. He also shows that $\sigma(\lambda) = \infty$ if $V(r) = cr^{-\mu}$ near infinity with $\mu \leqslant 2$. Jauch and Sinha (1972) use trace methods to prove finiteness for non-singular potentials decreasing like $r^{-3-\epsilon}$ ($\epsilon > 0$). Their approach is applicable to abstract scattering systems. Some further publications dealing with properties of the on-shell S-matrix are cited in the review by Martin and Misra (1974) on the Kato–Rosenblum lemma and its relation to high-energy scattering. An asymptotic expansion for $\bar{\sigma}(\lambda)$ for smooth potentials based on the eikonal approximation was obtained by Hunziker (1963).

This paper is organised as follows: In § 2 we collect the necessary definitions from scattering theory and the properties of Hilbert–Schmidt operators that we shall need. In § 3 we explain our approach and derive simple bounds on the total cross section. Section 4 is devoted to an improvement of these bounds in order to obtain a stronger result on the high-energy behaviour of the cross section. Finally in § 5 we mention some possible generalisations, including potential scattering with a relativistic free Hamiltonian, and point out a few more general properties of cross sections. We shall essentially follow the notation of Amrein *et al* (1977).

2. Physical and mathematical preliminaries

We consider the complex Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3)$ of all absolutely square-integrable functions defined on $\mathbb{R}^3$, with scalar product $(f, g) = \int d^3x\, f^*(x) g(x)$ and norm $\|f\| = \sqrt{(f, f)}$. Let Σ be a measurable subset of $\mathbb{R}^3$. For $f \in L^2(\mathbb{R}^3)$, the number

$$\frac{1}{\|f\|^2} \int_\Sigma |f(\boldsymbol{x})|^2\, d^3x$$

is interpreted as the probability (in the state f) of finding a particle localised in Σ (i.e. multiplication by x_k is the kth component Q_k of the position operator, $k = 1, 2, 3$.) We also denote by $(\mathscr{F}f)(\boldsymbol{k})$ or $\tilde{f}(\boldsymbol{k})$ the value of the Fourier transform $\tilde{f}$ of f at the point $\boldsymbol{k} \in \mathbb{R}^3$. Let H_0 be the usual free Hamiltonian, given formally by $H_0 = -\Delta$ (we set the mass equal to $\frac{1}{2}$ and $\hbar = 1$). H_0 is the self-adjoint operator defined by

$$(\tilde{H}_0 f)(\boldsymbol{k}) = |\boldsymbol{k}|^2 \tilde{f}(\boldsymbol{k}) \tag{3}$$

with domain

$$D(H_0) = \left[f \in L^2(\mathbb{R}^3) \,\middle|\, \int \||\boldsymbol{k}|^2 \tilde{f}(\boldsymbol{k})|^2\, d^3k < \infty \right]. \tag{4}$$

The total Hamiltonian H is a self-adjoint operator given formally as $H = H_0 + V$. If the (real-valued) potential function V is bounded or square-integrable over $\mathbb{R}^3$, the operator sum $H_0 + V$, defined on $D(H_0)$, is self-adjoint. If V has strong local singularities, H will be a self-adjoint extension of the operator $-\Delta + V(\boldsymbol{x})$, which is defined on some (not necessarily dense) linear subset $\mathscr{M}$ of $L^2(\mathbb{R}^3)$.

The free time-evolution is given by the (strongly) continuous one-parameter group $\{U_t\}$, $-\infty < t < \infty$, of unitary operators defined as $U_t = \exp(-\mathrm{i}H_0 t)$. Similarly the total evolution is given by a continuous unitary one-parameter group $\{V_t\}$, where $V_t = \exp(-\mathrm{i}Ht)$. The wave operators Ω_+ and Ω_- are defined as (A^* denoting the adjoint of A):

$$\Omega_+ = s\text{-}\lim_{t\to+\infty} V_t^* U_t, \qquad \Omega_- = s\text{-}\lim_{t\to-\infty} V_t^* U_t, \tag{5}$$

if the limits exist. They are strong limits in the Hilbert space, meaning for instance that for each $f \in \mathscr{H}$, $\|\Omega_+ f - V_t^* U_t f\| \to 0$ as $t \to +\infty$. $\Omega_\pm$ are isometric operators, i.e. $\|\Omega_\pm f\| = \|f\|$ for all f, or equivalently $\Omega_\pm^* \Omega_\pm = I$. The scattering operator is defined as

$$S = \Omega_+^* \Omega_-. \tag{6}$$

For the physical interpretation of these operators, we refer to chapter 4 of Amrein *et al* (1977). $\Omega_\pm$ intertwine V_t and U_t, whereas S commutes with U_t; i.e. for all real t

$$V_t \Omega_+ = \Omega_+ U_t, \qquad V_t \Omega_- = \Omega_- U_t \tag{7}$$

and

$$U_t S = S U_t. \tag{8}$$

In order to treat potentials with strong local singularities, the following observation is very useful. Let f be any state vector in $L^2(\mathbb{R}^3)$. If the time $|t|$ is very large, then the state $U_t f$, obtained by letting f evolve under the free time evolution, has very small probability of being localised in a bounded region of configuration space. In other words free particles propagate to infinity as $t \to \pm\infty$. A precise formulation is as follows. Consider the ball $S_K = \{\boldsymbol{x} \mid |\boldsymbol{x}| \leq K\}$. Then, for each fixed $K < \infty$ and each $f \in \mathscr{H}$

$$\lim_{t\to\pm\infty} \int_{S_K} |(U_t f)(\boldsymbol{x})|^2 \, \mathrm{d}^3x = 0. \tag{9}$$

More generally, let ϕ be an infinitely differentiable function from $\mathbb{R}^3$ to $\mathbb{R}$ such that $\phi(\boldsymbol{x}) = 0$ if $|\boldsymbol{x}| \leq K$, $\phi(\boldsymbol{x}) = 1$ if $|\boldsymbol{x}| \geq K'$, where K and $K' > K$ are any fixed positive numbers. We shall also denote by ϕ the multiplication operator by the function $\phi(\boldsymbol{x})$, i.e. $(\phi f)(\boldsymbol{x}) = \phi(\boldsymbol{x}) f(\boldsymbol{x})$. Then for each $f \in L^2(\mathbb{R}^3)$

$$\lim_{t\to\pm\infty} \|(I - \phi) U_t f\|^2 \equiv \lim_{t\to\pm\infty} \int_{|\boldsymbol{x}|\leq K'} |1 - \phi(\boldsymbol{x})|^2 |(U_t f)(\boldsymbol{x})|^2 \, \mathrm{d}^3x = 0. \tag{10}$$

For a proof, see Amrein *et al* (1977), remark 3.15 or corollary 7.8.

It follows from (10) that $\|V_t^*(I - \phi) U_t f\| \to 0$ as $t \to \pm\infty$ for each $f \in \mathscr{H}$, i.e. $s\text{-}\lim V_t^*(I - \phi) U_t = 0$ as $t \to \pm\infty$. Hence

$$\begin{aligned} \Omega_\pm &= s\text{-}\lim_{t\to\pm\infty} V_t^* U_t = s\text{-}\lim_{t\to\pm\infty} V_t^*(I - \phi) U_t + s\text{-}\lim_{t\to\pm\infty} V_t^* \phi U_t \\ &= s\text{-}\lim_{t\to\pm\infty} V_t^* \phi U_t. \end{aligned} \tag{11}$$

This last equation allows one to eliminate completely the local part of the potential in certain considerations in scattering theory, for instance in the existence proof of the wave operators. In fact the wave operators exist provided that $V(\boldsymbol{x})$ verifies certain hypotheses for large $|\boldsymbol{x}|$ (e.g. tending to zero faster than a Coulomb potential), irrespective of its behaviour in a finite region. This result is due to Kupsch and Sandhas (1966) (it is also given in Amrein *et al* (1977), proposition 8.31). An alternative proof of it will follow from our estimates in § 3.

Equation (8), which expresses the conservation of the kinetic energy in the scattering process, allows one to diagonalise simultaneously the free Hamiltonian H_0 (the infinitesimal generator of the group $\{U_t\}$) and the scattering operator S. We first diagonalise H_0 by introducing its spectral representation. For this one uses spherical polar coordinates $(\lambda, \boldsymbol{\omega})$ in momentum space, where $\lambda = \boldsymbol{k}^2$ and $\boldsymbol{\omega} = (\theta, \phi)$ is a vector on the unit sphere $S^{(2)} = \{\boldsymbol{k} | \boldsymbol{k}^2 = 1\}$. Let $f \in L^2(\mathbb{R}^3)$. For each fixed $\lambda > 0$, let

$$f_\lambda(\omega) = \frac{1}{\sqrt{2}} \lambda^{1/4} \tilde{f}(\sqrt{\lambda}\boldsymbol{\omega}). \tag{12}$$

A short computation using the Parseval identity in $L^2(\mathbb{R}^3)$ shows that

$$\|f\|^2 = \int |\tilde{f}(\boldsymbol{k})|^2 \, \mathrm{d}^3 k = \int_0^\infty \mathrm{d}\lambda \int_{S^{(2)}} \mathrm{d}\omega |f_\lambda(\boldsymbol{\omega})|^2, \tag{13}$$

where $\mathrm{d}\omega = -\mathrm{d}\cos\theta \, \mathrm{d}\phi$. Hence for almost all $\lambda > 0$, the function f_λ (as a function of $\boldsymbol{\omega}$) is square-integrable over $S^{(2)}$, i.e. belongs to $\mathcal{H}_0 \equiv L^2(S^{(2)})$. The scalar product in $L^2(S^{(2)})$ is

$$(f, g)_0 = \int_{S^{(2)}} \mathrm{d}\omega \, f^*(\boldsymbol{\omega}) g(\boldsymbol{\omega}). \tag{14}$$

From the definition (3) of H_0, one sees that $(H_0 f)_\lambda(\boldsymbol{\omega}) = \lambda f_\lambda(\boldsymbol{\omega})$. In other words the correspondence $f \mapsto \{f_\lambda\}_{\lambda>0}$ defines a unitary map $\mathcal{U}_0$ from $L^2(\mathbb{R}^3)$ onto $L^2((0, \infty); \mathcal{H}_0)$ (the Hilbert space of measurable functions from $(0, \infty)$ to $\mathcal{H}_0$ with norm (13)), such that $\mathcal{U}_0 H_0 \mathcal{U}_0^{-1}$ is just multiplication by λ (Amrein *et al* 1977 p 225).

Since S commutes with H_0, it must be diagonal in $L^2((0, \infty); \mathcal{H}_0)$. This means that for each $\lambda > 0$, there exists an operator $S(\lambda)$ acting in $\mathcal{H}_0 = L^2(S^{(2)})$ such that for all λ

$$(\mathcal{U}_0 S \mathcal{U}_0^{-1} f)_\lambda = S(\lambda) f_\lambda. \tag{15}$$

$S(\lambda)$ is called the *on-shell S-matrix* at energy λ. It is useful to define also $R = S - I$. Clearly R also commutes with H_0, so that

$$(\mathcal{U}_0 R \mathcal{U}_0^{-1} f)_\lambda = R(\lambda) f_\lambda, \qquad \text{with } R(\lambda) = S(\lambda) - I_0 \dagger \tag{16}$$

where I_0 is the identity operator in $\mathcal{H}_0$ (Amrein *et al* 1977 § 5.7). $(-2\pi\mathrm{i})^{-1} R(\lambda)$ is often called the T-matrix.

To relate the S-operator to the scattering cross section, one considers the scattering of a beam of uncorrelated particles by the potential V, (see Amrein *et al* 1977 § 7.3). The beam is mathematically described as an ensemble of states $\{g_{\boldsymbol{a}}\}$. For this one starts from a fixed state g which is assumed to be a wave packet with almost sharp momentum $\boldsymbol{k}_0$. The states of the ensemble $\{g_{\boldsymbol{a}}\}$ are obtained by a translation of g in the impact parameter plane (the plane orthogonal to $\boldsymbol{k}_0$) by a vector $\boldsymbol{a} \in \mathbb{R}^2$ and by choosing a uniform distribution of the values of $\boldsymbol{a}$ over this plane. For a given fixed cone C in

† In perhaps more familiar language, if $R(\lambda)$ is an integral operator with kernel $R(\lambda; \boldsymbol{\omega}, \boldsymbol{\omega}')$, then the scattering operator S has kernel $\delta(\boldsymbol{k} - \boldsymbol{k}') + 2k^{-1} R(\lambda; \boldsymbol{\omega}, \boldsymbol{\omega}') \delta(\boldsymbol{k}^2 - \boldsymbol{k}'^2)$ in momentum space with respect to the measure $\mathrm{d}^3 k'$.

configuration space with apex at the origin, the cross section for scattering into C is then defined as the quotient of the number of particles scattered into C (i.e. the sum over all $\boldsymbol{a}$ of the probability that a particle with initial state $g_{\boldsymbol{a}}$ will be observed in C at $t=+\infty$) and the number of points $\boldsymbol{a}$ in a unit square of $\mathbb{R}^2$.

The derivation of a reasonable expression for the cross section involves the following two points: (i) Let $f \in L^2(\mathbb{R}^3)$ be any state and assume that scattering is initiated in the state f. Then the probability that at time $t=+\infty$ the particle be localised in a cone C in configuration space with apex at the origin is the same as the probability that the corresponding final state Sf have momentum in C:

$$\lim_{t\to+\infty}\int_C |(V_t\Omega_- f)(\boldsymbol{x})|^2\,\mathrm{d}^3x = \int_C |(\tilde{S}f)(\boldsymbol{k})|^2\,\mathrm{d}^3k. \tag{17}$$

This is the scattering into cones formula (Dollard 1969, 1973). It is valid if the range of Ω_- is contained in that of Ω_+ (see Amrein *et al* (1977) proposition 7.15) and will be generalised in § 5. (ii) Let Σ be an interval containing the energy support of the initial state g, i.e. such that $g_\lambda = 0$ for $\lambda \notin \Sigma$. Assume that for each $\lambda \in \Sigma$, $R(\lambda)$ is a Hilbert–Schmidt operator (see below for properties of Hilbert–Schmidt operators) and that

$$\int_\Sigma \lambda^{-1}\|R(\lambda)\|^2_{\mathrm{HS}}\,\mathrm{d}\lambda < \infty \tag{18}$$

where the Hilbert–Schmidt norm is with respect to $L^2(S^{(2)})$. Since each Hilbert–Schmidt operator in an L^2-space is an integral operator with square-integrable kernel, $R(\lambda)$ will have a kernel $R(\lambda;\boldsymbol{\omega},\boldsymbol{\omega}')$ such that

$$(R(\lambda)f_\lambda)(\boldsymbol{\omega}) = \int_{S^{(2)}} R(\lambda;\boldsymbol{\omega},\boldsymbol{\omega}')f_\lambda(\boldsymbol{\omega}')\,\mathrm{d}\omega' \tag{19}$$

and

$$\int_{S^{(2)}}\mathrm{d}\omega\int_{S^{(2)}}\mathrm{d}\omega'\,|R(\lambda;\boldsymbol{\omega},\boldsymbol{\omega}')|^2 = \|R(\lambda)\|^2_{\mathrm{HS}} < \infty. \tag{20}$$

Under the assumption (18) one can compute explicitly the ensemble average and prove that the scattering amplitude for scattering from the initial direction $\boldsymbol{\omega}_0$ into the final direction $\boldsymbol{\omega}$ at energy λ is nothing but

$$f(\lambda;\boldsymbol{\omega}_0\to\boldsymbol{\omega}) = -2\pi\mathrm{i}\lambda^{-1/2}R(\lambda;\boldsymbol{\omega},\boldsymbol{\omega}_0). \tag{21}$$

The details of this simple proof may be found in § 7.3 of Amrein *et al* (1977). Moreover, the total scattering cross section, averaged over all initial directions, is simply given by (Amrein *et al* (1977) § 7.4)

$$\bar{\sigma}(\lambda) \equiv \frac{1}{4\pi}\int_{S^{(2)}}\mathrm{d}\omega_0\int_{S^{(2)}}\mathrm{d}\omega\,|f(\lambda;\boldsymbol{\omega}_0\to\boldsymbol{\omega})|^2 = \pi\lambda^{-1}\|R(\lambda)\|^2_{\mathrm{HS}}. \tag{22}$$

In Amrein *et al* 1977 it is shown by stationary methods that $\bar{\sigma}(\lambda)$ is finite and a continuous function of λ if e.g. V is square-integrable over some ball S_M and verifies $|V(\boldsymbol{x})| \leq (1+|\boldsymbol{x}|)^{-2-\epsilon}$ $(\epsilon > 0)$ for $|\boldsymbol{x}| > M$. In the next section we give a simple time-dependent proof of (18) for potentials verifying essentially the same hypothesis near infinity but having almost arbitrary local singularities.

We now assemble the necessary results about Hilbert–Schmidt operators. Their proofs are quite elementary (e.g. Ringrose 1971, § 2.4). Let $\mathscr{H}$ be a separable Hilbert space. The class of all Hilbert–Schmidt operators on $\mathscr{H}$ will be denoted by $\mathscr{B}_2(\mathscr{H})$ or simply by $\mathscr{B}_2$. An everywhere-defined bounded linear operator A in $\mathscr{H}$ is said to be a Hilbert–Schmidt operator if

$$\|A\|_{\mathrm{HS}}^2 \equiv \sum_k \|Ae_k\|^2 < \infty \tag{23}$$

where $\{e_k\}$ is an orthonormal basis of $\mathscr{H}$ (the sum in (23) is the same for each orthonormal basis). $\mathscr{B}_2$ is itself a Hilbert space with scalar product

$$(A, B)_2 = \mathrm{Tr}\, A^*B, \qquad (A, B \in \mathscr{B}_2) \tag{24}$$

where Tr denotes the trace in $\mathscr{H}$, i.e. $\mathrm{Tr}\, C = \Sigma_k (e_k, Ce_k)$. $\|\cdot\|_{HS}$ is the norm in $\mathscr{B}_2$, i.e.

$$\|A\|_{\mathrm{HS}}^2 = (A, A)_2 = \mathrm{Tr}\, A^*A. \tag{25}$$

The Schwarz inequality in $\mathscr{B}_2$ means that

$$\mathrm{Tr}\, A^*B = (A, B)_2 \leq [(A, A)_2 (B, B)_2]^{1/2} = \|A\|_{\mathrm{HS}} \|B\|_{\mathrm{HS}}. \tag{26}$$

One also has

$$\|B\|_{\mathrm{HS}} = \|B^*\|_{\mathrm{HS}} \tag{27}$$

and

$$\|AB\|_{\mathrm{HS}} \leq \|A\|\, \|B\|_{\mathrm{HS}} \tag{28}$$

where A is any bounded everywhere-defined linear operator and $\|A\|$ is its operator norm.

If $\{A_n\}$ is a sequence of Hilbert–Schmidt operators converging to A in Hilbert–Schmidt norm, i.e. if $\|A_n - A\|_{\mathrm{HS}} \to 0$ as $n \to \infty$, then A_n also converges strongly to A, i.e. $\|A_n f - Af\| \to 0$ as $n \to \infty$ for each $f \in \mathscr{H}$ (to see this, it suffices to take a basis in (23) such that $e_1 = f$).

Finally, if $\mathscr{H} = L^2(\mathbb{R}^n)$ or some other L^2-space, then A is a Hilbert–Schmidt operator if and only if it is an integral operator with square-integrable kernel. We shall use this in momentum space: if

$$(\tilde{A}f)(\boldsymbol{k}) = \int a(\boldsymbol{k}, \boldsymbol{k}')\tilde{f}(\boldsymbol{k}')\, \mathrm{d}^n k', \tag{29}$$

then

$$\|A\|_{\mathrm{HS}}^2 = \int\int |a(\boldsymbol{k}, \boldsymbol{k}')|^2\, \mathrm{d}^n k\, \mathrm{d}^n k'. \tag{30}$$

3. Finiteness of the total cross section

Given any real $\rho \in L^2(0, \infty)$ such that $\|\rho\| = 1$, we can define a linear operator $P(\rho)$ from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ by

$$(\mathscr{U}_0 P(\rho) f)_\lambda = \rho(\lambda) \int_0^\infty \rho(\mu)(\mathscr{U}_0 f)_\mu\, \mathrm{d}\mu. \tag{31}$$

The above is an integral of a vector-valued function in $L^2(S^{(2)})$. Such integrals may be defined either by approximating Riemann sums (if the integrand is strongly continuous) or more generally in terms of the scalar product with a fixed vector in $L^2(S^{(2)})$ (Amrein *et* *al* (1977) § 4.4). It is not difficult to see that $P(\rho)$ is the orthogonal projection operator onto the subspace of $L^2(\mathbb{R}^3)$ consisting of those f such that $(\mathcal{U}_0 f)_\lambda$ has the form of a product of $\rho(\lambda)$ with some vector in $L^2(S^{(2)})$. Such f 'factorise' in momentum space.

There is a close connection between the Hilbert–Schmidt norm of $RP(\rho)$ (acting in $L^2(\mathbb{R})^3$) and the Hilbert–Schmidt norms of the $R(\lambda)$ (acting in $L^2(S^{(2)})$.) In fact, given an orthonormal basis $\{e_k\}$ of $L^2(S^{(2)})$, $\{\mathcal{U}_0^{-1}(\rho(\lambda)e_k)\}$ is an orthonormal basis of the range of $P(\rho)$, so that

$$\|RP(\rho)\|_{\mathrm{HS}}^2 = \sum_k \|R\mathcal{U}_0^{-1}(\rho(\lambda)e_k)\|^2. \tag{32}$$

Using (13) and (16) this gives

$$\|RP(\rho)\|_{\mathrm{HS}}^2 = \int_0^\infty \mathrm{d}\lambda \sum_k \|R(\lambda)\rho(\lambda)e_k\|_0^2 = \int_0^\infty \mathrm{d}\lambda\ (\rho(\lambda))^2\|R(\lambda)\|_{\mathrm{HS}}^2. \tag{33}$$

Thus, in order to prove (18), we have only to show that $RP(\rho)$ is Hilbert–Schmidt for some suitable function ρ. Moreover, if we can do this for a sufficiently wide class of functions ρ we can deduce the finiteness of $\|R(\lambda)\|_{\mathrm{HS}}$ for almost all energies λ, and through (22) the finiteness of the total cross section, for a suitable class of potentials.

Although it might appear more natural to take ρ to be, say, the characteristic function of some bounded interval, our method of estimating $\|RP(\rho)\|_{\mathrm{HS}}$ is such that it is often more convenient to take ρ to be a smooth differentiable function.

From (6), using the isometry of Ω_+, we have

$$R = \Omega_+^*\Omega_- - I = \Omega_+^*(\Omega_- - \Omega_+).$$

Hence

$$\|RP(\rho)\|_{\mathrm{HS}} \leq \|(\Omega_+ - \Omega_-)P(\rho)\|_{\mathrm{HS}}. \tag{34}$$

Introducing as in § 2, equation (11), the operator of multiplication in position space by some smooth function ϕ which vanishes at the singularities of the potential V (i.e. according to (1) we have $\phi(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| \leq M$), it follows that

$$(\Omega_+ - \Omega_-)P(\rho) = \int_{-\infty}^\infty \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}(V_t^*\phi U_t P(\rho)). \tag{35}$$

We take $\phi = I$ if $M = 0$ in (1). Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_t^*\phi U_t P(\rho)) = \mathrm{i} V_t^*(H\phi - \phi H_0)U_t P(\rho), \tag{36}$$

by using the triangle inequality for norms (Amrein *et al* (1977) proposition 4.12), we have

$$\|RP(\rho)\|_{\mathrm{HS}} \leq \int_{-\infty}^\infty \mathrm{d}t\ \|(H\phi - \phi H_0)U_t P(\rho)\|_{\mathrm{HS}}. \tag{37}$$

(For ρ decreasing sufficiently rapidly as $\lambda \to \infty$, one has range $(P(\rho)) \subseteq D(H_0)$. Moreover we take $D(H)$ sufficiently large that $\phi D(H_0) \subseteq D(H)$, which is possible under the hypothesis (1), (Pearson 1975b). For more general ρ, one may proceed formally and justify the final results by a limiting argument.)

If the integral on the RHS of (37) is convergent, then (i) we have $\int_0^\infty \mathrm{d}\lambda\ \lambda(\rho(\lambda))^2\bar{\sigma}(\lambda) < \infty$, so that $\bar{\sigma}(\lambda)$ is finite for almost all λ in the support of ρ; (ii) by considering the integral over the positive and negative real line respectively, we may deduce (see Amrein *et al* (1977), proposition 4.13) the convergence of $(V_t^*\phi U_t-\phi)P(\rho)$ to $(\Omega_\pm-\phi)P(\rho)$ in Hilbert–Schmidt norm. In particular the wave operators exist (as strong limits) on the range of $P(\rho)$. If the ranges of the $P(\rho)$ (for various ρ) span a dense subset of $L^2(\mathbb{R}^3)$ we can deduce (see Amrein *et al* (1977) proposition 2.17) the existence of $\Omega_\pm$ on $L^2(\mathbb{R}^3)$. (For this we need the ρ to span a dense subset of $L^2(0,\infty)$, which will hold if, e.g., ρ can be any infinitely-differentiable function with compact support in $(0,\infty)$.)

It remains, then, to estimate integrals of the type (37), and to do this we rely on the following.

Lemma 1. Let $\rho(\lambda)$ be a smooth function vanishing in a neighbourhood of $\lambda=0$, and let W be an operator of multiplication in position space by some real function $W(\boldsymbol{x})$. Then

(i)

$$\lambda^{1/4}\rho\in L^2(0,\infty),\qquad W\in L^2(\mathbb{R}^3)\Rightarrow\int_{-\infty}^{\infty}\mathrm{d}t\,\|WU_tP(\rho)\|_{\mathrm{HS}}^2=\frac{1}{2\pi}\|\lambda^{1/4}\rho\|^2\|W\|^2. \tag{38}$$

(ii)

$$\lambda^{-1/4}\rho\in L^2(0,\infty),\qquad \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^{1/4}\rho)\in L^2(0,\infty),\qquad (1+|\boldsymbol{x}|)W\in L^2(\mathbb{R}^3)\Rightarrow$$

$$\int_{-\infty}^{\infty}\mathrm{d}t\,t^2\|WU_tP(\rho)\|_{\mathrm{HS}}^2=\frac{1}{2\pi}\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^{1/4}\rho)\right\|^2\|W\|^2+\frac{1}{24\pi}\|\lambda^{-1/4}\rho\|^2\||\boldsymbol{x}|W\|^2. \tag{39}$$

Proof. Taking adjoints, we have $\|WU_tP(\rho)\|_{\mathrm{HS}}=\|P(\rho)U_t^*W\|_{\mathrm{HS}}$. Now in momentum space U_t^*W is an integral operator having kernel $(2\pi)^{-3/2}\exp(\mathrm{i}k^2t)\tilde{W}(\boldsymbol{k}-\boldsymbol{k}')$. Using (12) and (31) we find that $P(\rho)U_t^*W$ is an integral operator with kernel

$$(2\pi)^{-3/2}\lambda^{-1/4}\rho(\lambda)\int_0^{\infty}\mathrm{d}\mu\,\mu^{1/4}\rho(\mu)\,\mathrm{e}^{\mathrm{i}\mu t}\tilde{W}(\sqrt{\mu}\boldsymbol{\omega}-\boldsymbol{k}')$$

where $\boldsymbol{k}=\sqrt{\lambda}\boldsymbol{\omega}$. Writing $\mathrm{d}^3k=k^2\,\mathrm{d}k\,\mathrm{d}\omega=\frac{1}{2}\lambda^{1/2}\,\mathrm{d}\lambda\,\mathrm{d}\omega$ and using $\|\rho\|=1$, we can evaluate the Hilbert–Schmidt norm, and changing finally the integration variable from μ to λ gives

$$\|WU_tP(\rho)\|_{\mathrm{HS}}^2=\tfrac{1}{2}(2\pi)^{-3}\int\mathrm{d}\omega\,\mathrm{d}^3k'\left|\int_0^{\infty}\mathrm{d}\lambda\,\lambda^{1/4}\rho(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda t}\tilde{W}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')\right|^2. \tag{40}$$

In (i) of the lemma, Parseval's identity gives

$$\int_{-\infty}^{\infty}\mathrm{d}t\,\|WU_tP(\rho)\|_{\mathrm{HS}}^2=\tfrac{1}{2}(2\pi)^{-2}\int\mathrm{d}\lambda\,\mathrm{d}\omega\,\mathrm{d}^3k'|\lambda^{1/4}\rho(\lambda)\tilde{W}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')|^2, \tag{41}$$

and (38) follows if we first carry out the integration d^3k'.

For (ii) of the lemma, first integrate by parts in (40), giving

$$\int_0^{\infty}\mathrm{d}\lambda\,\lambda^{1/4}\rho(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda t}\,\tilde{W}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')$$

$$=\frac{\mathrm{i}}{t}\int_0^{\infty}\mathrm{d}\lambda\,\mathrm{e}^{\mathrm{i}\lambda t}\left\{\frac{\mathrm{d}}{\mathrm{d}\lambda}[\lambda^{1/4}\rho(\lambda)]\tilde{W}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')+\lambda^{1/4}\rho(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\tilde{W}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')\right\}. \tag{42}$$

In the second term of the RHS we can write $\mathrm{d}/\mathrm{d}\lambda = \frac{1}{2}\lambda^{-1/2}\,\mathrm{d}/\mathrm{d}k$ and

$$\frac{\mathrm{d}\tilde{W}}{\mathrm{d}k}(\boldsymbol{k}-\boldsymbol{k}') = -\mathrm{i}\,\tilde{W}_{\omega}(\boldsymbol{k}-\boldsymbol{k}')$$

with

$$\tilde{W}_{\omega}(\boldsymbol{p}) = \boldsymbol{\omega}\,.\,[\mathscr{F}(\boldsymbol{x}W(\boldsymbol{x}))](\boldsymbol{p}).$$

Parseval's identity now gives

$$\int_0^\infty \mathrm{d}t\, t^2 \|WU_tP(\rho)\|_{\mathrm{HS}}^2$$

$$= \tfrac{1}{2}(2\pi)^{-2}\int \mathrm{d}\lambda\ \mathrm{d}\omega\ \mathrm{d}^3k' \left|\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^{1/4}\rho(\lambda))\,\tilde{W}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}') - \frac{\mathrm{i}}{2}\lambda^{-1/4}\rho(\lambda)\,\tilde{W}_{\omega}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')\right|^2. \qquad (43)$$

Integrating first d^3k' and using the identity (a, b real)

$$\int \mathrm{d}\omega\ \|aW + \mathrm{i}bW_{\omega}\|^2 = 4\pi\left(a^2\|W\|^2 + \frac{b^2}{3}\||\boldsymbol{x}|W\|^2\right),$$

(39) follows and the lemma is proven. □

Remark 1. The integration by parts, with zero boundary contribution from $\lambda = \infty$, may be justified rigorously e.g. for the 'cut-off' function $\chi_{S_m}W$ which has support in the ball $|\boldsymbol{x}| \leq m$. (In that case the Fourier transform is infinitely differentiable and bounded.) Equation (39) may then be extended to general W by letting the cut-off radius m tend to infinity.

Remark 2. Since both U_t and $P(\rho)$ are rotation invariant, it follows that $\|WU_tP(\rho)\|_{\mathrm{HS}}^2 = \|W_{\mathrm{rot}}U_tP(\rho)\|_{\mathrm{HS}}^2$, where W_{rot} is obtained from W by rotation about some axis through the origin. If we now average over such rotations we see that $\|WU_tP(\rho)\|_{\mathrm{HS}}^2 = \|W_sU_tP(\rho)\|_{\mathrm{HS}}^2$, where now $W_s(|\boldsymbol{x}|)$ is a spherically symmetric function defined by $W_s^2 = (4\pi)^{-1}\int W^2\,\mathrm{d}\omega$. So in the proof of lemma 1 we could have assumed spherical symmetry without loss of generality.

Remark 3. Because of the term $\mathrm{d}\rho/\mathrm{d}\lambda$ on the RHS, the integral in (39) will diverge in the limiting case where ρ is the characteristic function of a finite interval. This is purely a consequence of our method of estimation, and once we have proved the convergence of the integral in (33) for some suitable set of smooth functions ρ the integral will also converge for characteristic functions.

In order to estimate the contribution in (37) of the commutator of H_0 with ϕ we need also to consider $\|W\boldsymbol{P}\,.\,\boldsymbol{n}U_tP(\rho)\|_{\mathrm{HS}}$, where $\boldsymbol{P}$ is the momentum operator and $\boldsymbol{n}$ is a fixed unit vector. In that case the corresponding equation to (40) has a factor $\lambda^{1/2}\boldsymbol{\omega}\,.\,\boldsymbol{n}$ in the integrand with respect to λ. For fixed $\boldsymbol{a}$, $\boldsymbol{b}$, one may verify that

$$\int (\boldsymbol{\omega}\,.\,\boldsymbol{a})^2(\boldsymbol{\omega}\,.\,\boldsymbol{b})^2\,\mathrm{d}\omega = \frac{4\pi}{15}(|\boldsymbol{a}|^2|\boldsymbol{b}|^2 + 2|\boldsymbol{a}\,.\,\boldsymbol{b}|^2). \qquad (44)$$

(Alternatively, use the obvious bound $4\pi|\boldsymbol{a}|^2|\boldsymbol{b}|^2$ for this integral if no exact expression is required). Hence one obtains

Lemma 2.

(i) $\lambda^{3/4}\rho \in L^2(0,\infty), \qquad W \in L^2(\mathbb{R}^3) \Rightarrow$

$$\int_{-\infty}^{\infty} \mathrm{d}t\, \|W\boldsymbol{P}\,.\,\boldsymbol{n}U_tP(\rho))\|_{\mathrm{HS}}^2 = \frac{1}{6\pi}\|\lambda^{3/4}\rho\|^2\,\|W\|^2. \tag{45}$$

(ii) $\lambda^{1/4}\rho \in L^2(0,\infty), \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^{3/4}\rho) \in L^2(0,\infty), \qquad (1+|\boldsymbol{x}|)W \in L^2(\mathbb{R}^3) \Rightarrow$

$$\int_{-\infty}^{\infty} \mathrm{d}t\, t^2\|W\boldsymbol{P}\,.\,\boldsymbol{n}U_tP(\rho)\|_{\mathrm{HS}}^2$$

$$= \frac{1}{120\pi}\|\lambda^{1/4}\rho\|^2\{\||\boldsymbol{x}|\,W\|^2 + 2\|\boldsymbol{x}\,.\,\boldsymbol{n}W\|^2\} + \frac{1}{6\pi}\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^{3/4}\rho)\right\|^2\|W\|^2. \tag{46}$$

In (37), let us now write

$$H\phi - \phi H_0 = V\phi + [H_0, \phi] = V\phi - (\Delta\phi) - 2i(\nabla\phi)\,.\,\boldsymbol{P} \tag{47}$$

where, e.g., $\Delta\phi$ refers to the operator of multiplication by $(\Delta\phi)(\boldsymbol{x})$. Then lemmas 1 and 2 may already be used to prove finiteness of the total cross section for potentials V satisfying

$$\int_{|\boldsymbol{x}|>M} \mathrm{d}^3x\,(1+|\boldsymbol{x}|)^2|V(\boldsymbol{x})|^2 < \infty. \tag{48}$$

For such a potential, let $W = -\Delta\phi + V\phi$, and choose ρ suitably. Then (38) and (39) imply

$$\int_{-\infty}^{\infty} \mathrm{d}t\,\|(-\Delta\Phi + V\Phi)U_tP(\rho)\|_{\mathrm{HS}}^2 < \infty$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}t\, t^2\|(-\Delta\phi + V\phi)U_tP(\rho)\|_{\mathrm{HS}}^2 < \infty.$$

(Note that $\Delta\phi$ has compact support.) Hence

$$\int_{-\infty}^{\infty} \mathrm{d}t(1+t^2)\|(-\Delta\phi + V\phi)U_tP(\rho)\|_{\mathrm{HS}}^2 < \infty,$$

so that by Schwarz's inequality we find

$$\int_{-\infty}^{\infty} \mathrm{d}t\,\|(-\Delta\phi + V\phi)U_tP(\rho)\|_{\mathrm{HS}}$$

$$\leqslant \left(\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(1+t^2)}\right)^{1/2}\left(\int_{-\infty}^{\infty} \mathrm{d}t\,(1+t^2)\|(-\Delta\phi + V\phi)U_tP(\rho)\|_{\mathrm{HS}}^2\right)^{1/2} < \infty. \tag{49}$$

Similarly, taking $W=\partial\phi/\partial x_k$ and $\boldsymbol{n}$ the unit vector along the k'th coordinate axis, lemma 2 may be used to show that

$$\int_{-\infty}^{\infty} \mathrm{d}t \, \|(\boldsymbol{\nabla}\phi)\,.\,\boldsymbol{P}U_tP(\rho)\|_{\mathrm{HS}}<\infty$$

so that the integral on the RHS of (37) is convergent, and we have proved finiteness of the total cross section for this class of potentials at almost all energies. (The class of permissible functions ρ is quite large and certainly includes infinitely differentiable functions having compact support in $(0, \infty)$.)

The conditions here imposed on the potential would not allow V to decay at infinity more slowly than $|\boldsymbol{x}|^{-5/2}$. We know, however, that the cross section should be finite with a decay like $|\boldsymbol{x}|^{-2-\epsilon}$ $(\epsilon>0)$, and to prove this we shall need the rather more refined estimates given by:

Lemma 3. Let ρ satisfy the conditions of (i) and (ii) in lemma 1, and suppose that $(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon}W\in L^2(\mathbb{R}^3)$ for some $\epsilon>0$. Then there exist $\delta>0$ and $c(\rho)>0$ such that

$$\int_{-\infty}^{\infty} \mathrm{d}t \, (1+|t|)^{1+\delta}\|WU_tP(\rho)\|_{\mathrm{HS}}^2\leq c(\rho)\|(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon}W\|^2. \tag{50}$$

Proof. Let $W_0\in L^2\,(\mathbb{R}^3)$, and define

$$G_\alpha(t)=\|(1+|\boldsymbol{x}|)^{-\alpha}W_0U_tP(\rho)\|_{\mathrm{HS}}. \tag{51}$$

Then lemma 1 implies

$$\int_{-\infty}^{\infty} \mathrm{d}t \, (G_0(t))^2\leq c_1(\rho)\|W_0\|^2 \tag{52}$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}t \, (1+|t|)^2(G_1(t))^2\leq c_2(\rho)\|W_0\|^2, \tag{53}$$

where $c_1(\rho)$, $c_2(\rho)$ may be calculated explicitly from (38) and (39). Now by equations (25)–(27),

$$\begin{aligned}(G_{(\alpha+\beta)/2}(t))^2&=\mathrm{Tr}\{P(\rho)U_t^*W_0(1+|\boldsymbol{x}|)^{-(\alpha+\beta)}W_0U_tP(\rho)\}\\&\leq\|P(\rho)U_t^*W_0(1+|\boldsymbol{x}|)^{-\alpha}\|_{\mathrm{HS}}\times\|(1+|\boldsymbol{x}|)^{-\beta}W_0U_tP(\rho)\|_{\mathrm{HS}}\\&=G_\alpha(t)G_\beta(t).\end{aligned} \tag{54}$$

Hence if

$$\int_{-\infty}^{\infty} \mathrm{d}t \, (1+|t|)^p(G_\alpha(t))^2<\infty \tag{55}$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}t \, (1+|t|)^q(G_\beta(t))^2<\infty, \tag{56}$$

we can deduce, by Schwarz's inequality,

$$\int_{-\infty}^{\infty} \mathrm{d}t\,(1+|t|)^{(p+q)/2}(G_{(\alpha+\beta)/2}(t))^2$$

$$\leqslant \int_{-\infty}^{\infty} \mathrm{d}t\,\{(1+|t|)^{p/2}G_\alpha(t)\}\{(1+|t|)^{q/2}G_\beta(t)\}$$

$$\leqslant \left(\int_{-\infty}^{\infty} \mathrm{d}t\,(1+|t|)^{p}(G_\alpha(t))^2\right)^{1/2}\left(\int_{-\infty}^{\infty} \mathrm{d}t\,(1+|t|)^{q}(G_\beta(t))^2\right)^{1/2} < \infty. \tag{57}$$

Applying (57) to (52) and (53) with $\alpha=0$, $\beta=1$, $p=0$, $q=2$, we have

$$\int_{-\infty}^{\infty} \mathrm{d}t\,(1+|t|)(G_{\frac{1}{2}}(t))^2 \leqslant (c_1c_2)^{1/2}\|W_0\|^2. \tag{58}$$

Applying (57) to (53) and (58) gives

$$\int_{-\infty}^{\infty} \mathrm{d}t\,(1+|t|)^{1+\frac{1}{2}}(G_{\frac{1}{2}+\frac{1}{4}}(t))^2 \leqslant c_1^{1/4}c_2^{3/4}\|W_0\|^2. \tag{59}$$

We can now combine this inequality with (58) and continue in this way, to obtain

$$\int_{-\infty}^{\infty} \mathrm{d}t\,(1+|t|)^{1+2^{-n}}(G_{\frac{1}{2}+2^{-n-1}}(t))^2 \leqslant c_1^{\zeta_n}c_2^{\xi_n}\|W_0\|^2, \tag{60}$$

with $\zeta_n=\frac{1}{2}-2^{-n-1}$, $\xi_n=\frac{1}{2}+2^{-n-1}$ and $n=1,2,3,\ldots$. Taking n sufficiently large that $2^{-n-1}<\epsilon$, we have in that case $G_{\frac{1}{2}+\epsilon}\leqslant G_{\frac{1}{2}+2^{-n-1}}$, so that with $\delta=2^{-n}$ (60) implies

$$\int_{-\infty}^{\infty} \mathrm{d}t\,(|1+|t|)^{1+\delta}(G_{\frac{1}{2}+\epsilon}(t))^2 \leqslant c_1^{\zeta_n}c_2^{\xi_n}\|W_0\|^2. \tag{61}$$

Equation (50) now follows on writing $W_0=(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon}W$.

Note that, if $\epsilon=2^{-n-1}$, we can take $\delta=2\epsilon$, in which case in (50) $c=c_1^{\frac{1}{2}-\epsilon}c_2^{\frac{1}{2}+\epsilon}$, where $c_1(\rho)$, $c_2(\rho)$ are the constants appearing in (52), (53). □

By analogy with lemma 2 we also have:

Lemma 4. Let ρ satisfy the conditions of (i) and (ii) in lemma 2, and suppose that $(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon}W\in L^2(\mathbb{R}^3)$ for some $\epsilon>0$. Then there exist $\delta>0$ and $c'(\rho)>0$ such that

$$\int_{-\infty}^{\infty} \mathrm{d}t\,(1+|t|)^{1+\delta}\|W\boldsymbol{P}\cdot\boldsymbol{n}U_tP(\rho)\|_{\mathrm{HS}}^2 \leqslant c'(\rho)\|(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon}W\|^2. \tag{62}$$

Proof. Let $W_0\in L^2(\mathbb{R}^3)$, and define

$$F_\alpha(t)=\|(1+|\boldsymbol{x}|)^{-\alpha}W_0\boldsymbol{P}\cdot\boldsymbol{n}U_tP(\rho)\|_{\mathrm{HS}}.$$

Then lemma 2 implies

$$\int_{-\infty}^{\infty} \mathrm{d}t\,(F_0(t))^2 \leqslant c_1'(\rho)\|W_0\|^2 \tag{63}$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}t\,(1+|t|)^2(F_1(t))^2 \leqslant c_2'(\rho)\|W_0\|^2. \tag{64}$$

Now proceed as in the proof of lemma 3. Note that in this case with $\epsilon = 2^{-n-1}$, $\delta = 2\epsilon$, we have $c' = (c_1')^{\frac{1}{2}-\epsilon}(c_2')^{\frac{1}{2}+\epsilon}$. □

We are now ready to state a result which implies in particular the finiteness of the total cross section for potentials behaving at large distances like $(1+|\boldsymbol{x}|)^{-2-\epsilon}$, but with arbitrary local singularities.

Theorem 1. Suppose the potential V satisfies (1) with $\upsilon = \frac{1}{2}+\epsilon$ for some $M \geqslant 0$ and $\epsilon > 0$. Then the wave operators $\Omega_\pm$ exist, and $\bar{\sigma}(\lambda)$ is finite for almost all λ.

(*Remark.* To be precise, one should assert the finiteness of $\|R(\lambda)\|_{\mathrm{HS}}^2$ for almost all λ rather than that of $\bar{\sigma}(\lambda)$. As pointed out in § 2, the identification of $\bar{\sigma}(\lambda)$ with $\pi\lambda^{-1}\|R(\lambda)\|_{\mathrm{HS}}^2$ requires the validity of the scattering into cones formula, the proof of which needs a hypothesis on the location of the singularities of V in the region $|\boldsymbol{x}| \leqslant M$. This point will be taken up in § 5.4.)

Proof. (cf argument following lemma 2). We have only to estimate the RHS of (37), using (47). From (50) of lemma 3,

$$\int_{-\infty}^{\infty} \mathrm{d}t\ (1+|t|)^{1+\delta}\|(-\Delta\phi + V\phi)U_tP(\rho)\|_{\mathrm{HS}}^2 < \infty,$$

so that

$$\int_{-\infty}^{\infty} \mathrm{d}t\ \|(-\Delta\phi + V\phi)U_tP(\rho)\|_{\mathrm{HS}}$$

$$\leqslant \left(\int \frac{\mathrm{d}t}{(1+|t|)^{1+\delta}}\right)^{1/2}\left(\int_{-\infty}^{\infty} \mathrm{d}t\ (1+|t|)^{1+\delta}\|(-\Delta\phi + V\phi)U_tP(\rho)\|_{\mathrm{HS}}^2\right)^{1/2} < \infty.$$

In the same way, the contribution to the RHS of (37) of the $(\boldsymbol{\nabla}\phi\,.\,\boldsymbol{P})$ term in (47) may be estimated using (62). Hence from (37) we can deduce $\|RP(\rho)\|_{\mathrm{HS}} < \infty$. The conclusions of the theorem now follow, since the class of permissible functions ρ is sufficiently wide for the ranges of the $P(\rho)$'s to generate the entire Hilbert space. □

4. Bounds on the cross section and high-energy behaviour

The estimates that we have derived in § 3 to prove finiteness of $\bar{\sigma}(\lambda)$ may at the same time be used to obtain precise upper bounds. These are not *pointwise* bounds, but rather bounds on some integral of the cross section over some range of energies; they may be used for example, to give an explicit upper bound, for a given potential, to the average cross section over some given finite range of energies.

Consider first the case of a potential satisfying

$$\int \mathrm{d}^3x\ (1+|\boldsymbol{x}|)^{1+2\epsilon}|V(\boldsymbol{x})|^2 < \infty \tag{65}$$

for some $\epsilon > 0$ (referred to below as the non-singular case). By a straightforward application of our estimates, using the method of interpolation of lemma 3, we find the following bound:

$$\int_0^\infty \lambda(\rho(\lambda))^2\bar{\sigma}(\lambda)\,\mathrm{d}\lambda \leqslant A_\epsilon(\rho)\|(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon}V\|^2, \tag{66}$$

where

$$A_\epsilon(\rho)=\frac{1}{2}\epsilon^{-1}\|\lambda^{1/4}\rho\|^{1-2\epsilon}\left(\frac{1}{6}\|\lambda^{-1/4}\rho\|^2+2\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^{1/4}\rho)\right\|^2+2\|\lambda^{1/4}\rho\|^2\right)^{\frac{1}{2}+\epsilon} \tag{67}$$

and $0<\epsilon\leqslant\frac{1}{2}$. We emphasise that in (67) we have made no attempt to obtain the optimal function $A_\epsilon(\rho)$. In deriving (67) we have actually assumed that ϵ is some integral multiple of 2^{-n-1} for some positive integer n; this assumption has no practical significance.

Notice that it is inherent in our method of estimation that, although the LHS of (66) is invariant under translations $V(\boldsymbol{x})\to V(\boldsymbol{x}+\boldsymbol{a})$, the RHS is not. Therefore in applying (66), a suitable choice of origin in position space must be made. Both sides of (66) are, however, invariant under *rotations* about the origin.

In the case of a potential which may be singular in some finite region $|\boldsymbol{x}|<M$, but which satisfies (1) for some $\upsilon>\frac{1}{2}$, we have to estimate $\|(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon}\Delta\phi\|$ and $\|(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon}\boldsymbol{\nabla}\phi\|$ where ϕ is the function appearing in (37). A simple choice is to take $\phi(\boldsymbol{x})=f(|\boldsymbol{x}|/M)$ if $M\geqslant 1$ and $\phi(\boldsymbol{x})=f(|\boldsymbol{x}|)$ if $M\leqslant 1$, where $f(r)$ is a smooth non-decreasing function satisfying $f(r)=0$ for $0\leqslant r\leqslant 1$ and $f(r)=1$ for $r\geqslant 2$. In this case we find from lemmas 3 and 4, for this class of singular potentials,

$$\begin{aligned}\int_0^\infty &\lambda(\rho(\lambda)^2\bar{\sigma}(\lambda)\,\mathrm{d}\lambda\\ &\leqslant A_\epsilon(\rho)\|(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon}\chi_{[M,\infty)}V\|^2+A_\nu(\rho)(A_1+B_1M^\nu)\\ &\quad+A_\nu n(\lambda^{1/2}\rho)(A_2+B_2M^{2+\nu})\end{aligned} \tag{68}$$

where A_1, B_1, A_2, B_2 are constants (depending on the choice of f), $A_\nu(\rho)$ (and hence $A_\nu(\lambda^{1/2}\rho)$) is given again by (67), ν is any number in $(0,\frac{1}{2}]$ and $\chi_{[M,\infty)}$ is the characteristic function of the region $|\boldsymbol{x}|\geqslant M$. Again we regard (68) as a rather crude estimate. However, interesting conclusions can already be drawn from (66) and (68).

Observe that, for potentials V of finite range which have support contained in the region $|\boldsymbol{x}|<M$, the bound (68) is independent of the potential (and, in particular, of the coupling constant) but, for given $\rho(\lambda)$, depends only on M. For *large M* the bound is like $M^{2+\nu}$, where ν may be taken arbitrarily small (classically one has πM^2.) For *small M*, the bound *does not* vanish in the limit as $M\to 0$. In fact the bound cannot vanish in the limit $M\to 0$, since there are potentials with arbitrarily small support for which the cross section is bounded away from zero (Newton 1966 § 14.1).

The preceding result may also be applied in the opposite way by remarking that, given the size of the cross section, one may deduce a lower bound on the range M of the interaction. An optical determination of this lower bound would require a special analysis of the bounds of § 3 by varying the functions ρ and ϕ. The result also justifies to some extent the classical picture that the total cross section is a measure of the size of the region of interaction in relative coordinates between the two colliding particles. In this context it is interesting to remark that the finiteness of the total cross section is quite independent of the explicit form of the free Hamiltonian but depends only on V (see also Martin and Misra 1973). H_0 may for instance be any increasing function of $\boldsymbol{P}^2$. If V

verifies (65), then one obtains a bound of the form (66), with $A_\epsilon(\rho)$ given by an expression similar to (67), see § 5.1. The dependence of the bound on H_0 appears only in the explicit form of $A_\epsilon(\rho)$.

We may also use (66) and (68) to estimate the high-energy behaviour of the cross sections. Giving $\rho(\lambda)$ a simple power behaviour as $\lambda \to \infty$ we easily find that, for $\lambda_0 > 0$ and any $\theta > 0$,

$$\int_{\lambda_0}^{\infty} \frac{d\lambda \bar{\sigma}(\lambda)}{\lambda^{\frac{1}{2}+\theta}} < \infty \tag{69}$$

in the non-singular case, and

$$\int_{\lambda_0}^{\infty} \frac{d\lambda \bar{\sigma}(\lambda)}{\lambda^{\frac{3}{2}+\theta}} < \infty \tag{69'}$$

in the singular case. This suggests that local singularities may have an effect on the high-energy behaviour of cross sections, which may decay to zero more slowly than in the non-singular case, or even not tend to zero. However, (69) does not give the best known results, since $\bar{\sigma}(\lambda)$ should decay in this case like $1/\lambda$ (Amrein *et al* 1977, chapter 12) whereas (69) corresponds to a decay roughly like $\lambda^{-\frac{1}{2}+\epsilon}$ $(\epsilon > 0)$.

It seems worthwhile, therefore, to extend some of our results to obtain something near the 'best possible' high-energy behaviour of cross sections. It will be convenient to introduce the notation, for $W \in L^2(\mathbb{R}^3)$,

$$|||W|||_\theta^2 = \|(1+|\boldsymbol{k}|)^{\frac{1}{2}-\theta}\tilde{W}(\boldsymbol{k})\|^2 + \|(1+|\boldsymbol{k}|)^{\frac{1}{2}-\theta}\tilde{W}_1(\boldsymbol{k})\|^2 + \|(1+|\boldsymbol{k}|)^{\frac{1}{2}-\theta}|\nabla \tilde{W}_1(\boldsymbol{k})|\|^2 \tag{70}$$

where $\theta > 0$, $W_1(\boldsymbol{x}) = (1+|\boldsymbol{x}|)^{-1}W(\boldsymbol{x})$ and ∇ denotes differentiation with respect to the components of $\boldsymbol{k}$, so that $\boldsymbol{n}\,.\,\nabla W(\boldsymbol{k})$ is just the Fourier transform of $-\mathrm{i}(\boldsymbol{n}\,.\,\boldsymbol{x})W(\boldsymbol{x})$.

Our high-energy estimates of cross sections are based on:

Lemma 5. Let ρ be a smooth function vanishing near $\lambda = 0$ and satisfying $\rho(\lambda) = \lambda^{-(\frac{1}{2}+\epsilon)}$ for large λ, with $\epsilon > 0$. Suppose $|||W|||_\theta < \infty$ for all $\theta > 0$. Then for some $\delta > 0$, $\theta > 0$

(i) $$\int_{-\infty}^{\infty} dt\, \|(H_0+1)^{-\frac{1}{4}+\delta}WU_tP(\rho)\|_{\mathrm{HS}}^2 \leq \text{const.}|||W|||_\theta^2 \tag{71}$$

(ii) $$\int_{-1}^{+1} dt\, |t|^{-\delta}\|(H_0+1)^{-\frac{1}{4}-\delta}WU_tP(\rho)\|_{\mathrm{HS}}^2 \leq \text{const.}|||W|||_\theta^2 \tag{72}$$

(iii) $$\int_{-\infty}^{\infty} dt\, t^2\|(H_0+1)^{\frac{1}{4}-\delta}W_1U_tP(\rho)\|_{\mathrm{HS}}^2 \leq \text{const.}|||W|||_\theta^2 \tag{73}$$

with W_1 given as above.

Proof. (i) Following the derivation of (41) we have

$$\int_{-\infty}^{\infty} dt\|(H_0+1)^{-\frac{1}{4}+\delta}WU_tP(\rho)\|_{\mathrm{HS}}^2$$

$$= \frac{1}{2(2\pi)^2}\int d\lambda\; d\omega\; d^3k'|\lambda^{1/4}\rho(\lambda)\tilde{W}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')(|\boldsymbol{k}'|^2+1)^{-\frac{1}{4}+\delta}|^2$$

$$= \frac{1}{(2\pi)^2}\int d^3k\; d^3k'(\rho(k^2))^2|\tilde{W}(\boldsymbol{k}-\boldsymbol{k}')|^2(\boldsymbol{k}'^2+1)^{-\frac{1}{2}+2\delta}$$

on writing $\lambda = k^2$, $\mathrm{d}\lambda\ \mathrm{d}\omega = 2k^{-1}\ \mathrm{d}^3k$. Making the change of variable $\boldsymbol{k}' = \boldsymbol{k} - \boldsymbol{k}''$ we have

$$\frac{1}{(2\pi)^2}\int \mathrm{d}^2k\ \mathrm{d}^3k''\,(\rho(k^2))^2|\tilde{W}(\boldsymbol{k}'')|^2((\boldsymbol{k}-\boldsymbol{k}'')^2+1)^{-\frac{1}{2}+2\delta}. \tag{74}$$

We may divide the region of integration with respect to $\boldsymbol{k}$ into two regions, respectively $\{|\boldsymbol{k}-\boldsymbol{k}''| > \frac{1}{2}|\boldsymbol{k}|\}$ and $\{|\boldsymbol{k}-\boldsymbol{k}''| < \frac{1}{2}|\boldsymbol{k}|\}$.

Over the first of these regions, our assumptions on $\rho(\lambda)$ imply that $\int \mathrm{d}^3k(\rho(k^2))^2((\frac{1}{2}|\boldsymbol{k}|)^2+1)^{-\frac{1}{2}+2\delta} < \infty$ provided $\delta < \epsilon$, in which case the double integral in (74) is bounded by const.$\|W\|^2$. In the second of these regions, $|\boldsymbol{k}| > \frac{2}{3}|\boldsymbol{k}''|$, so that $(\rho(k^2))^2 = \mathrm{O}(|\boldsymbol{k}''|^{-(2+4\epsilon)})$. Moreover, the integration region over $\boldsymbol{k}$ has volume $\mathrm{O}(|\boldsymbol{k}''|^3)$, so that the integral d^3k is $\mathrm{O}(|\boldsymbol{k}''|^{1-4\epsilon})$ as $|\boldsymbol{k}''| \to \infty$, and (71) holds with $\theta = 2\epsilon$.

(ii) In order to prove (72), let us write, following (40),

$$\int_{-1}^{+1} \mathrm{d}t\ |t|^{-\delta}\|(H_0+1)^{-\frac{1}{4}+\delta}WU_tP(\rho)\|^2_{\mathrm{HS}}$$

$$= \frac{1}{2(2\pi)^3}\int \mathrm{d}\omega\ \mathrm{d}^3k'\ \mathrm{d}\lambda\ \mathrm{d}\mu\ \mathrm{d}t\,\lambda^{1/4}\rho(\lambda)\mu^{1/4}\rho(\mu)\mathrm{e}^{\mathrm{i}(\lambda-\mu)t}|t|^{-\delta}$$
$$\times \tilde{W}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')\tilde{W}^*(\sqrt{\mu}\boldsymbol{\omega}-\boldsymbol{k}')(\boldsymbol{k}'^2+1)^{-\frac{1}{2}+2\delta}. \tag{75}$$

We carry out first the integration with respect to t, thus making use of the identity:

$$\int \mathrm{d}t\,(\mathscr{F}^{-1}f)^*(t)q(t)(\mathscr{F}^{-1}f)(t) = \frac{1}{(2\pi)^{1/2}}\int f^*(\mu)\tilde{q}(\mu-\lambda)f(\lambda)\,\mathrm{d}\lambda\ \mathrm{d}\mu \tag{76}$$

in the case $f(\lambda) = \lambda^{1/4}\rho(\lambda)\tilde{W}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')$. (76) holds, for $q \geq 0$, $f \in L^2(\mathbb{R})$, provided the RHS is absolutely convergent, as we shall show in this case. (Note that in proving (i) we have shown $f \in L^2$ for almost all $\boldsymbol{k}'$ and $\boldsymbol{\omega}$.) Here $\tilde{q}(\lambda)$ is the Fourier transform of $\chi(t)|t|^{-\delta}$, where χ is the characteristic function of the interval $[-1, 1]$. It is not difficult to show, e.g. by a change of integration variable $z = \lambda t$, that

$$\tilde{q}(\lambda = 0(|\lambda|^{-1+\delta}) \qquad \text{as } |\lambda| \to \infty \tag{77}$$

whereas $\tilde{q}(\lambda)$ is locally bounded.

After integrating with respect to t, an application of Schwarz's inequality to (75) leads to the bound

$$\text{const.}\left[\int \mathrm{d}\omega\ \mathrm{d}^3k'\ \mathrm{d}\lambda\ \mathrm{d}\mu\,\mu^{1/2}(\rho(\mu))^2|\tilde{q}(\mu-\lambda)||\tilde{W}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')|^2(\boldsymbol{k}'^2+1)^{-\frac{1}{2}+2\delta}\right]^{1/2}$$
$$\times\left[\int \mathrm{d}\omega\ \mathrm{d}^3k'\ \mathrm{d}\lambda\ \mathrm{d}\mu\,\lambda^{1/2}(\rho(\lambda))^2|\tilde{q}(\mu-\lambda)||\tilde{W}(\sqrt{\mu}\boldsymbol{\omega}-\boldsymbol{k}')|^2\right.$$
$$\left.\times(\boldsymbol{k}'^2+1)^{-\frac{1}{2}+2\delta}\right]^{1/2}. \tag{78}$$

The two factors are identical (on interchanging λ, μ). We next integrate over μ in the first factor. For this we divide the integration region with respect to μ into the two regions $\{|\mu-\lambda| \geq \frac{1}{2}\lambda\}$ and $\{|\mu-\lambda| \leq \frac{1}{2}\lambda\}$. In the first region we also have $|\mu-\lambda| \geq \frac{1}{3}\mu$, so that from (77) we have:

$$|\tilde{q}(\mu-\lambda)| \leqslant \frac{\text{const.}}{(1+|\mu-\lambda|)^{1-\delta}} \leqslant \frac{\text{const.}}{(1+\frac{1}{3}\mu)^{\frac{1}{4}-2\epsilon+\beta}(1+\frac{1}{2}\lambda_2)^{\frac{3}{4}+2\epsilon-\delta-\beta}}$$

for any $\beta > 0$.

Since $\int \mathrm{d}\mu\, \mu^{1/2}\rho(\mu)^2(1+\frac{1}{3}\mu)^{-\frac{1}{2}+2\epsilon-\beta} < \infty$, the integral over the first region is bounded by const.$(1+\frac{1}{2}\lambda)^{-\frac{1}{2}-2\epsilon+\delta+\beta}$. In the second region, $\frac{1}{2}\lambda \leqslant \mu \leqslant \frac{3}{2}\lambda$, and the μ integration is bounded by

$$\text{const. } \lambda^{-\frac{1}{2}-2\epsilon} \int_{\frac{1}{2}\lambda}^{\frac{3}{2}\lambda} \mathrm{d}\mu\ (1+|\mu-\lambda|)^{-1+\delta}$$

which is $O(\lambda^{-\frac{1}{2}-2\epsilon+\delta})$ as $\lambda \to \infty$. We have, then, finally to estimate

$$\int \mathrm{d}\lambda\ \mathrm{d}\omega\ \mathrm{d}^3k'\ (1+\lambda)^{-\frac{1}{2}-2\epsilon+\delta+\beta}|\tilde{W}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')|^2(\boldsymbol{k}'^2+1)^{-\frac{1}{2}+2\delta}.$$

But this is precisely the estimate we have already made in proving (71) above, except that now we must take $(\rho(\lambda))^2 = \lambda^{-1-2\epsilon+\delta+\beta}$ for large λ. We find that (72) holds, with $\delta < \frac{2}{3}\epsilon$, if $\theta < 2\epsilon - \delta$.

(iii) Following the derivation of (43), we have

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}t\ &\|(H_0+1)^{\frac{1}{4}-\delta}W_1U_tP(\rho)\|_{\mathrm{HS}}^2 \\ &\leqslant \text{const.} \int \mathrm{d}\lambda\ \mathrm{d}\omega\ \mathrm{d}^3k'\ |\lambda^{-1/4}\rho(\lambda)\tilde{W}_{1,\omega}(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')|^2(\boldsymbol{k}'^2+1)^{\frac{1}{2}-2\delta} \\ &\quad + \text{const.} \int \mathrm{d}\lambda\ \mathrm{d}\omega\ \mathrm{d}^3k' \left|\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^{1/4}\rho(\lambda))\tilde{W}_1(\sqrt{\lambda}\boldsymbol{\omega}-\boldsymbol{k}')\right|^2(\boldsymbol{k}'^2+1)^{\frac{1}{2}-2\delta}.\end{aligned}$$

Following the same arguments as at the beginning of (i), and making use of the inequality (Amrein *et al* (1977), lemma 16.12)

$$((\boldsymbol{k}-\boldsymbol{k}'')^2+1)^{\frac{1}{2}-2\delta} \leqslant \text{const.}(\boldsymbol{k}^2+1)^{\frac{1}{2}-2\delta}(\boldsymbol{k}''^2+1)^{\frac{1}{2}-2\delta}$$

in the equation replacing (74), we readily find that (73) holds with $\theta = 2\delta$. □

Remark. In lemma 5, one may for instance take $\delta = \frac{1}{2}\epsilon$, $\theta = \epsilon$, in which case the constants appearing in (71)–(73) depend only on ρ.

Following the derivation of (58) we can use (72) and (73) to show that

$$\int_{-1}^{1} \mathrm{d}t\ |t|^{1-\delta/2}(G_{1/2}(t))^2 \leqslant \text{const.}|||W_0|||_\theta^2 \tag{79}$$

where $G_\alpha(t)$ and W_0 are defined as in the proof of lemma 3 and we retain the assumption $\rho(\lambda) = \lambda^{-(1/2+\epsilon)}$ for large λ. An application of Schwarz's inequality now gives:

$$\left(\int_{-1}^{1} \mathrm{d}t\ |G_{1/2}(t)|\right)^2 \leqslant \text{const.} \int_{-1}^{1} |t|^{-1+\delta/2}\ \mathrm{d}t\ |||W_0|||_\theta^2, \tag{80}$$

so that certainly

$$\int_{-1}^{1} \mathrm{d}t\, |G_{\frac{1}{2}+\epsilon'}(t)| \leqslant \text{const.} |||W_0|||_\theta \tag{81}$$

for any $\epsilon' > 0$.

Following again the derivation of (58) we can use (71) and (73) to show that

$$\int_{-\infty}^{\infty} \mathrm{d}t\, |t| (G_{1/2}(t))^2 \leqslant \text{const.} |||W_0|||_\theta^2. \tag{82}$$

Since $\|W_0\| \leqslant |||W_0|||_\theta$ for $\theta < \frac{1}{2}$, (39) of lemma 1 implies

$$\int_{-\infty}^{\infty} \mathrm{d}t\, t^2 (G_1(t))^2 \leqslant \text{const.} |||W_0|||_\theta^2. \tag{83}$$

Following the derivation of (61) we can use (82) and (83) to show that

$$\int_{-\infty}^{\infty} \mathrm{d}t\, |t|^{1+\delta} (G_{\frac{1}{2}+\epsilon'}(t))^2 \leqslant \text{const.} |||W_0|||_\theta^2 \tag{84}$$

where ϵ' is an integral multiple of 2^{-n-1} for some positive integer n and $\delta = 2\epsilon'$. An application of Schwarz's inequality now gives:

$$\int_{|t| \geqslant 1} \mathrm{d}t\, |G_{\frac{1}{2}+\epsilon'}(t)| \leqslant \text{const.} |||W_0|||_\theta$$

and combining this result with (81) we now have:

$$\int_{-\infty}^{\infty} \mathrm{d}t\, |G_{\frac{1}{2}+\epsilon'}(t)| \leqslant \text{const.} |||W_0|||_\theta. \tag{85}$$

In other words,

$$|||(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon'} V|||_\theta < \infty \Rightarrow \int_{-\infty}^{\infty} \mathrm{d}t\, \|VU_t P(\rho)\|_{\mathrm{HS}} < \infty$$

and we can state:

Theorem 2. Suppose the potential V satisfies $|||(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon'} V|||_\theta < \infty$ for some $\epsilon' > 0$ (strictly for $\epsilon' = m2^{-n-1}$) and for all $\theta > 0$, where $|||\cdot|||_\theta$ is given by (70). Then, for all $\epsilon > 0$, and θ sufficiently small,

$$\int_{1}^{\infty} \mathrm{d}\lambda\, \lambda^{-2\epsilon} \bar{\sigma}(\lambda) \leqslant \text{const.} |||(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon'} V|||_\theta < \infty. \tag{86}$$

Remark. For $W \in L^2(\mathbb{R}^3)$, the finiteness of $|||W|||_\theta$ imposes a bound on the behaviour of $\tilde{W}(\boldsymbol{k})$ for large $|\boldsymbol{k}|$. It may be checked that theorem 2 applies to potentials which are not too singular (e.g. the Yukawa potential) and which tend to zero as $|\boldsymbol{x}| \to \infty$ more rapidly than $|\boldsymbol{x}|^{-2-\epsilon}$, for some $\epsilon > 0$.

For potentials having arbitrary local singularities we have also to estimate in (37) the contributions from ϕ. The corresponding result to lemma 5 in that case, with $\rho(\lambda) = \lambda^{-(1+\epsilon)}$, follows in exactly the same way, with WU_t, W_1U_t in (71)–(73) replaced by $W\boldsymbol{P}\cdot\boldsymbol{n}U_t$, $W_1\boldsymbol{P}\cdot\boldsymbol{n}U_t$ respectively. We then have:

Theorem 3. Suppose the potential V satisfies $|||\chi_{[M,\infty)}(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon'}V|||_\theta<\infty$ for some $\epsilon'>0$ and all $\theta>0$, where $\chi_{[M,\infty)}$ is the characteristic function of the region $|\boldsymbol{x}|>M$. Then, for all $\epsilon>0$, and θ sufficiently small,

$$\int_1^\infty \mathrm{d}\lambda\ \lambda^{-1-\epsilon}\bar{\sigma}(\lambda)<\text{const.}|||\chi_{[M,\infty)}(1+|\boldsymbol{x}|)^{\frac{1}{2}+\epsilon'}V|||_\theta+F(M)<\infty. \tag{87}$$

A slight improvement of this result is mentioned in § 5.2.

5. Generalisations and comments

5.1. Further results on potential scattering

Our approach can be applied to the n-dimensional Schrödinger equation. The hypothesis on the potential V remains as equation (1) with $\upsilon>\frac{1}{2}$, where the integral is in $\mathbb{R}^n$ instead of $\mathbb{R}^3$. It essentially means that $V(\boldsymbol{x})$ must tend to zero faster than $r^{-\frac{1}{2}(n+1)-\epsilon}$ ($\epsilon>0$) as $r\to\infty$. Our estimates imply the existence of the wave operators and the finiteness of $\|R(\lambda)\|_{\mathrm{HS}}$ for almost all λ. Similarly one may prove the existence of $\Omega_\pm$ and the finiteness of $\|R(\lambda)\|_{\mathrm{HS}}$ for certain momentum–dependent potentials (for other proofs of the existence of $\Omega_\pm$ in this case, see Schechter 1976, Berthier and Collet 1977). A special case of particular importance is that of spin–orbit interactions (van Winter and Brascamp 1968 and Amrein *et al* (1977) § 11.2).

Another important application is to N-body potential scattering for which rigorous stationary results are very scarce. Our method leads to the finiteness of the total cross section $\bar{\sigma}_{\alpha\to\beta}(\lambda)$ and to bounds similar to those of § 4 for scattering from a two-body initial channel α to any final channel β, provided that the pair potentials verify equation (1) and that their singularities are positive. The details are given in a separate report (Amrein *et al* 1979).

For potentials that decrease to zero more slowly than r^{-2}, $\bar{\sigma}(\lambda)$ is in general infinite (Villarroel 1970). Under suitable assumptions on the derivatives of V, one can however prove finiteness of the cross section for scattering into any closed cone not containing the forward direction. This question will be dealt with in a forthcoming paper.

As pointed out in § 4, the class of potentials satisfying (65) leads to a finite total cross section for other free Hamiltonians that are functions of $\boldsymbol{P}^2$. In fact we have:

Theorem 4. Let $\{\Lambda_k\}$ be a finite or countable family of disjoint open intervals such that $[0,\infty)=\overline{\bigcup_k\Lambda_k}$. Let $F:(0,\infty)\to\mathbb{R}$ be an increasing function which is twice differentiable on each Λ_k, and let $H_0=F(\boldsymbol{P}^2)$. Assume V verifies (65), and let H be a self-adjoint extension of $F(\boldsymbol{P}^2)+V$ defined on $\mathscr{D}=\{f|\tilde{f}$ has compact support$\}$. Then the wave operators $\Omega_\pm=s\text{-lim}\exp(\mathrm{i}Ht)\exp(-\mathrm{i}H_0t)$ as $t\to\pm\infty$ exist and $R(\lambda)$ is Hilbert–Schmidt for almost all λ.

The hypothesis on F includes for instance the relativistic free Hamiltonian $H_0=(\boldsymbol{P}^2+m^2)^{1/2}$. If F has jump discontinuities at the end points of Λ_k, one has a free Hamiltonian whose spectrum consists of energy bands. Also notice that F may be a *bounded* function.

Proof. The diagonalisation of H_0 is carried out similarly to (12) by setting:

$$f_\lambda(\boldsymbol{\omega})=2^{-1/2}G(\lambda)\tilde{f}(L(\lambda)\boldsymbol{\omega}) \tag{88}$$

where $G(\lambda)=[F^{-1}(\lambda)]^{1/4}[F'(F^{-1}(\lambda))]^{-1/2}$, $L(\lambda)=[F^{-1}(\lambda)]^{1/2}$ and F^{-1} is the inverse function of F. (For $H_0=\boldsymbol{P}^2$: $F(x)=x$, $F^{-1}(x)=x$ and $F'(x)=1$, which gives (12).)

The proof of lemma 1 can be repeated, replacing $\lambda^{1/4}$ by $G(\lambda)$ and $\lambda^{1/2}$ by $L(\lambda)$ at the appropriate places and using $k^2\,\mathrm{d}k=\frac{1}{2}G(\lambda)^2\,\mathrm{d}\lambda$. One obtains:

$$\int_{-\infty}^{\infty}\mathrm{d}t\,\|W\,\mathrm{e}^{-\mathrm{i}F(\boldsymbol{P}^2)t}P(\rho)\|_{\mathrm{HS}}^2=\frac{1}{2\pi}\|G(\lambda)\rho\|^2\|W\|^2,\tag{89}$$

$$\int_{-\infty}^{\infty}\mathrm{d}t\,t^2\|W\,\mathrm{e}^{-\mathrm{i}F(\boldsymbol{P}^2)t}P(\rho)\|_{\mathrm{HS}}^2=\frac{1}{2\pi}\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}(G(\lambda)\rho)\right\|^2\|W\|^2$$

$$+\frac{1}{24\pi}\|[F^{-1}(\lambda)]^{-1/4}[F'(F^{-1}(\lambda))]^{-3/2}\rho\|^2\|\,|\boldsymbol{x}|W\|^2\tag{90}$$

where the support of ρ should be a subset of some Λ_k. One can now use these identities to interpolate exactly as in lemma 3 and prove the assertions of the theorem as in theorem 1. □

The scattering into cones formula for Hamiltonians of the above type has been established by Jauch *et al* (1972). The relation of the total cross section to $R(\lambda)$ is found as in § 7.3 of Amrein *et al* (1977) to be:

$$\bar{\sigma}(\lambda)=\frac{\pi}{|\boldsymbol{k}|^2}\|R(\lambda)\|_{\mathrm{HS}}^2,\qquad\text{where } |\boldsymbol{k}|^2=F^{-1}(\lambda).\tag{91}$$

Though the explicit form of the function F has no effect on the finiteness of $\bar{\sigma}(\lambda)$, it will influence the high-energy behaviour of the total cross section. For the relativistic free Hamiltonian we have $F(x)=(x+m^2)^{1/2}$, $F^{-1}(x)=x^2-m^2$ and $F'(x)=\frac{1}{2}(x+m^2)^{-1/2}$, and it turns out that all norms involving ρ in (89) and (90) are finite if e.g. $\rho(\lambda)=\lambda^{-3/2}(\lg\lambda)^{-\alpha}$ for $\lambda>2m^2$, with $\alpha>\frac{1}{2}$. Using this together with (91) in (33), one arrives at the following high-energy bound:

$$\int_{\lambda_0}^{\infty}\mathrm{d}\lambda\,\frac{\bar{\sigma}(\lambda)}{\lambda(\lg\lambda)^{1+\epsilon}}<\infty\text{ for each }\epsilon>0.\tag{92}$$

This is an improvement of the bound of Martin and Misra (1973) which was obtained by means of trace methods. Relation (92) excludes, for instance, a cross section such as $(\lg\lambda)^{\delta}$ as $\lambda\to\infty$, with $\delta>0$ (choose $\epsilon=\delta$ in (92) to get a contradiction). It is therefore also stronger than the bound of Froissart (1961). It must be borne in mind, though, that the latter is derived from different postulates and is valid under much more general circumstances than simply elastic single-channel potential scattering.

5.2. Further improvement of high-energy bounds

One can also improve the bounds in § 4 on the high-energy behaviour of cross sections by allowing $\rho(\lambda)$ to decrease as $\lambda^{-1/2}(\lg\lambda)^{-\beta}$ for large λ. (In that case one must replace $|t|^{-\delta}$ by $|\lg|t||^{q}$ in (72), with $q>1$, and introduce a similar factor into (73) for the integration over the interval $-1<t<1$. One can also take $\delta=0$ throughout.)

This gives rise to the result:

$$\int_{\lambda_0}^{\infty}\mathrm{d}\lambda\,\bar{\sigma}(\lambda)(\lg\lambda)^{-2\beta}<\infty\tag{93}$$

in the non-singular case, and with $\rho(\lambda)=\lambda^{-1}(\lg\lambda)^{-\beta}$ we find

$$\int_0^\infty d\lambda\,\frac{\bar{\sigma}(\lambda)(\lg\lambda)^{-2\beta}}{\lambda}<\infty \tag{94}$$

in the singular case, where $\beta>\frac{3}{2}$ in each case. In view of the hope (Frank *et al* 1971) that scattering theory with singular potentials should in some ways serve as a model of field theory, it is interesting that the second bound compares very closely with the Froissart bound $\bar{\sigma}(\lambda)=O((\lg\lambda)^2)$.

5.3. Bounds on the Born approximation

If $\Omega_\pm^{(B)}$ denotes the Born approximation to the wave operators in the case of a non-singular potential, equation (35) implies

$$(\Omega_+^{(B)}-\Omega_-^{(B)})P(\rho)=\mathrm{i}\int_{-\infty}^{\infty} dt\,(U_t^* VU_t)P(\rho). \tag{95}$$

It follows that, in this case, all of our estimates apply equally well to the Born approximation to the cross section as to the cross section itself.

5.4. Scattering into cones for singular potentials

The scattering-into-cones formula (17) is proved in Amrein *et al* (1977) under the hypothesis that $\Omega_-\mathscr{H}\subseteq\Omega_+\mathscr{H}$. This condition is satisfied if the theory is asymptotically complete, i.e. if $\Omega_-\mathscr{H}=\Omega_+\mathscr{H}=\mathscr{H}_{ac}(H)$ (the subspace of absolute continuity of H, which in many instances is equal to the orthogonal complement of the set of all eigenvectors of H). Asymptotic completeness is known to hold for non-singular potentials (Enss 1978 or Amrein *et al* (1977) chapters 9 and 10) as well as for a large class of singular potentials (see Amrein *et al* (1977) p 387 for references). An example of a potential that is singular at the origin and violates the condition $\Omega_-\mathscr{H}\subseteq\Omega_+\mathscr{H}$ was given by Pearson (1975a). A generalised theory of asymptotic completeness was then developed in Pearson (1975b). Assume that the singularities of V are restricted to a closed bounded subset N of $\mathbb{R}^3$ of Lebesgue measure zero, i.e.

$$\int_\Sigma d^3x\,|V(\boldsymbol{x})|^2<\infty \tag{96}$$

for all closed subsets Σ of the complement $\mathbb{R}^3\backslash N$ of N. Then Pearson (1975b) proves that each $g\in\mathscr{H}_{ac}(H)$ which is orthogonal to $\Omega_+\mathscr{H}$ has the property that

$$\lim_{t\to+\infty}\int_\Sigma d^3x\,|(V_t g)(\boldsymbol{x})|^2=0 \tag{97}$$

where Σ is as above. In other words, states in the (absolutely) continuous subspace of H that are orthogonal to all outgoing states have the property of being completely attracted by the singularities (i.e. localised in an arbitrarily small neighbourhood of N) as $t\to+\infty$.

This result allows us to generalise the scattering-into-cones formula to singular potentials satisfying (96). Let S_M be a ball in $\mathbb{R}^3$ containing N in its interior, $S'_M=\mathbb{R}^3\backslash S_M$ its complement. Let C be a cone with apex at the origin, and set $C_M=C\cap S'_M$. Let $f\in\mathscr{H}$ be an arbitrary initial state. Then the probability $P(f, C_M)$ that the corresponding scattering state $V_t\Omega_-f$ be localised in C_M at $t=+\infty$ is the same as the probability that the momentum of the outgoing part $Sf \equiv \Omega_+^*\Omega_- f$ of the final state lies in C:

$$P(f, C_M) = \lim_{t\to+\infty} \int_{C_M} d^3x \, |(V_t\Omega_- f)(\boldsymbol{x})|^2 = \int_C d^3k \, |\tilde{S}f)(\boldsymbol{k})|^2. \tag{98}$$

For the proof, one sets $\Omega_- f = g + h$ with $g = \Omega_+\Omega_+^*\Omega_- f$. g is the projection of $\Omega_- f$ onto the subspace $\Omega_+\mathscr{H}$ (the outgoing part of $\Omega_- f$) and h is orthogonal to this subspace (the absorbed part of $\Omega_- f$ (Amrein *et al* 1977 § 4.6).) Now equation (97) with $\Sigma = C_M$ implies that

$$\lim_{t\to+\infty} \int_{C_M} d^3x \, |(V_t h)(\boldsymbol{x})|^2 = 0.$$

Thus

$$\begin{aligned} P(f, C_M) &= \lim_{t\to+\infty} \int_{C_M} d^3x \, |(V_t\Omega_+\Omega_+^*\Omega_- f)(\boldsymbol{x})|^2 \\ &= \lim_{t\to+\infty} \int_{C_M} d^3x \, |(U_t Sf)(\boldsymbol{x})|^2 = \lim_{t\to+\infty} \int_C d^3x \, |(U_t Sf)(\boldsymbol{x})|^2 \\ &= \int_C d^3k \, |(\tilde{S}f)(\boldsymbol{k})|^2. \end{aligned}$$

The second identity follows from the definition of the wave operator, the third one from the fact that, under the free evolution U_t, all states are propagating to infinity, i.e. from equation (9), and the last identity is a simple property of the free evolution group (equation (3.51) of Amrein *et al* (1977)).

Once this generalised scattering-into-cones formula is established, we may assert the finiteness of $\bar{\sigma}(\lambda)$ for almost all λ under the hypothesis that:

$$\int_\Sigma d^3x \, (1+|\boldsymbol{x}|)^{2\upsilon} |V(\boldsymbol{x})|^2 < \infty \tag{99}$$

for some $\upsilon > \frac{1}{2}$ and all closed subsets Σ of $\mathbb{R}^3 \backslash N$.

If V has a hard core (say $V(\boldsymbol{x}) = +\infty$ for $|\boldsymbol{x}| \leq M$), is non-singular on S'_M and verifies (1), then one obtains similarly that $\bar{\sigma}(\lambda) < \infty$ for almost all λ by using the scattering theory for hard core potentials developed by Hunziker (1967) and the fact that there is no absorption (Amrein and Georgescu 1973).

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References

Amrein W O and Georgescu V 1973 *Helv. Phys. Acta* **46** 635

Amrein W O, Jauch J M and Sinha K B 1977 *Scattering Theory in Quantum Mechanics* (Reading, Mass.: Benjamin)

Amrein W O, Pearson D B and Sinha K B 1979 *Nuovo. Cim.* A to appear
Berthier A M and Collet P 1977 *J. Funct. Anal.* **26** 1
Deift P and Simon B 1976 *J. Funct. Anal.* **23** 218
Dollard J D 1969 *Comm. Math. Phys.* **12** 193
—— 1973 *J. Math. Phys.* **14** 708
Enss V 1977 *Comm. Math. Phys.* **52** 233
—— 1978 *Comm. Math. Phys.* **61** 285
Frank W M, Land D J and Spector R M 1971 *Rev. Mod. Phys.* **43** 36
Froissart M 1961 *Phys. Rev.* **123** 1053
Green T A and Lanford O E 1960 *J. Math. Phys.* **1** 139
Hunziker W 1963 *Helv. Phys. Acta* **36** 838
—— 1967 *Helv. Phys. Acta* **40** 1052
Jauch J M 1958 *Helv. Phys. Acta* **31** 127, 661
Jauch J M, Lavine R and Newton R G 1972 *Helv. Phys. Acta* **45** 220
Jauch J M and Sinha K B 1972 *Helv. Phys. Acta* **45** 580
Kupsch J and Sandhas W 1966 *Comm. Math. Phys.* **2** 147
Martin Ph and Misra B 1973 *J. Math. Phys.* **14** 997
—— 1974 *Scattering Theory in Mathematical Physics* ed. J A La Vita and J P Marchand (Dordrecht: Reidel) pp 173–92
Misra B, Speiser D and Targonski G 1963 *Helv. Phys. Acta* **36** 963
Newton R G 1966 *Scattering Theory of Waves and Particles* (New York: McGraw-Hill)
Pearson D B 1975a *Comm. Math. Phys.* **40** 125
—— 1975b *Helv. Phys. Acta* **48** 639
Ringrose J R 1971 *Compact Non-self-Adjoint Operators* (London: van Nostrand)
Schechter M 1976 *J. Math. Oxford Ser. (2)* **27** 111
Simon B 1977 *Comm. Math. Phys.* **55** 259
—— 1978 *Comm. Math. Phys.* **58** 205
Sinha K B 1977 *Ann. Inst. Henri Poincaré* A **26** 263
Villarroel D 1970 *Nuovo Cim.* A **70** 175
van Winter C and Brascamp H J 1968 *Comm. Math. Phys.* **11** 19